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FAST TRACK COMMUNICATION

Skyrmion multi-walls

J Silva Lobo and R S Ward

Department of Mathematical Sciences, Durham University, Durham DH1 3LE, UK

E-mail: j.i.silva-lobo@durham.ac.uk and richard.ward@durham.ac.uk

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Abstract
Skyrmion walls are topologically nontrivial solutions of the Skyrme system which are periodic in two spatial directions. We report numerical investigations which show that solutions representing parallel multi-walls exist. The most stable configuration is that of the square N-wall, which in the $N \to \infty$ limit becomes the cubically symmetric Skyrme crystal. There is also a solution resembling parallel hexagonal walls, but this is less stable.

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The Skyrme system, originally introduced as a model of nucleons, is an archetypal (3+1)-dimensional classical field theory admitting topological soliton solutions. Much is known about various types of skyrmion solutions, for example: isolated skyrmions in $\mathbb{R}^3$, up to relatively high charge [1–3]; a triply-periodic 'Skyrme crystal' [2, 4–6]; a doubly-periodic 'Skyrme domain wall' [7] and various types of singly-periodic 'Skyrme chains' [8].

The purpose of this communication is to investigate static N-wall solutions, i.e. the $N > 1$ generalization of the single-wall fields discussed in [7]. If one has two (or indeed N) well-separated parallel walls, then the force between them can be made attractive by a suitable relative orientation of the fields. So one expects there to be solutions representing N walls bound together, although *a priori* the walls might merge together to form a single wall.

We investigate this by numerical minimization of the energy, and our main findings are as follows. There are two obvious shapes for a single wall, namely square and hexagonal, and it is known [7] that the latter has slightly lower energy than the former. If walls are allowed to attract, then they do not merge but remain identifiable as separate parallel walls. There is a stable bound configuration representing two parallel hexagonal walls, but this is not the lowest energy 2-wall state. For $N \geqslant 2$, the lowest energy state consists of N parallel square walls (each one being a square array of half-skyrmions), and as $N \to \infty$ this approaches the Skyrme crystal.

The energy density of a static $SU(2)$-valued Skyrme field $U(x^j)$ on $\mathbb{R}^3$ is defined to be

$$\mathcal{E} := -\tfrac{1}{2}\,\mathrm{tr}(L_iL_i) - \tfrac{1}{16}\,\mathrm{tr}([L_i, L_j][L_i, L_j]), \qquad (1)$$

where $L_i = U^{-1}\partial U/\partial x^i$, and $x^j = (x, y, z)$ are the spatial coordinates. In what follows, let us write $U = \Phi_4 + \mathrm{i}\Phi_j\sigma_j$, where σ_j are the Pauli matrices, and the real 4-vector field $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ satisfies $\mathbf{\Phi}\cdot\mathbf{\Phi} = 1$.

In this communication, we deal with configurations which resemble N walls or sheets, each parallel to the xy-plane: so the field is periodic in x and y (with periods L_x and L_y, respectively) and satisfies the boundary condition

$$\Phi_4 \to \begin{cases} 1 & \text{as} \quad z \to -\infty, \\ (-1)^N & \text{as} \quad z \to \infty. \end{cases} \tag{2}$$

For $N = 1$, and more generally for N odd, one has a domain wall which separates two vacuum regions, where $\Phi_4 = 1$ and $\Phi_4 = -1$, respectively; for N even, one has the same vacuum on both sides of the multi-layered sheet. In the asymptotic region $|z| \gg 1$, the three fields Φ_j are small and they satisfy the Laplace equation, since the energy density reduces to $\mathcal{E} \approx (\partial_i\Phi_j)^2$. Assuming (without loss of generality) that $L_y \geqslant L_x$, we see by separating variables that the leading behaviour as $|z| \to \infty$ is typically $\Phi_j \approx C\sin(\mu y)\exp(-\mu|z|)$, where $\mu = 2\pi/L_y$. In particular, the fields approach their asymptotic values exponentially quickly, with a scale determined by the larger of L_x and L_y.

The topological charge Q (over a single cell) is

$$Q = \int_{T^2\times\mathbb{R}} \mathcal{Q}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{3}$$

where

$$\mathcal{Q} = \frac{1}{24\pi^2}\varepsilon_{ijk}\,\mathrm{tr}(L_iL_jL_k) \tag{4}$$

is the topological charge density. We claim that Q is an integer. If N is even, then (2) allows us to regard $\mathbf{\Phi}$ as being defined, for topological purposes, on $T^2 \times S^1$, and then Q equals the degree of $\mathbf{\Phi}$. If N is odd, then it is not quite so obvious why Q is an integer, but it follows from the theorem in the appendix of [8]. The energy E is defined to be

$$E := \frac{1}{12\pi^2}\int_{T^2\times\mathbb{R}} \mathcal{E}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{5}$$

and it satisfies the usual Faddeev bound $E \geqslant Q$.

In what follows, we describe N-wall configurations which were found by numerical minimization of the energy functional E. We used a first-order finite-difference scheme for E, with the spatial points (x, y, z) being represented by a rectangular lattice having lattice spacing h, and we applied conjugate-gradient minimization. The lattice error in E goes like h^2, and we extrapolated the finite-h results for both E and Q to $h = 0$. The extrapolated value of Q then gives a measure of the remaining error, which for the situations described below turns out to be less than 0.2%. The boundary condition (2) was modelled by imposing $\Phi_4 = 1$ at $z = -L_z/2$ and $\Phi_4 = (-1)^N$ at $z = L_z/2$. As remarked above, the walls are exponentially localized in z, and so as long as L_z is taken to be large enough, there is no discernable finite-size effect; a value of $L_z = 10+2N$ turns out to be sufficient for this. In each case, we adjusted the periods L_x and L_y to their optimal size, meaning that the energy-per-cell is made as small as possible. Numerical minima were randomly perturbed and then re-minimized, as a test of their stability. As initial configurations we used the same sort of 'rational map ansatz' as in [7], involving a Weierstrass elliptic function of $x + \mathrm{i}y$ (the lemniscatic form to get square symmetry, and the equianharmonic form to get hexagonal symmetry), together with a suitable profile function $f(z)$ satisfying $f(-L_z/2) = 0$ and $f(L_z/2) = N\pi$.

The results are consistent with the anticipated general principle that the lowest-energy configurations are arrays of half-skyrmions. For an N-wall, we expect that each fundamental

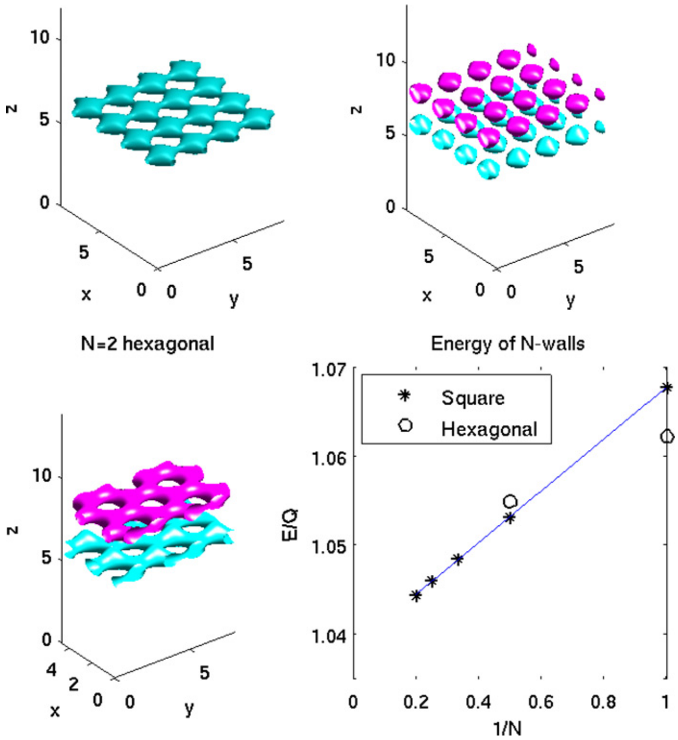

Figure 1. Energy densities of the square 1-wall, square 2-wall and hexagonal 2-wall, and plot of the energy $\widehat{E}$ for the square N-wall ($1 \leqslant N \leqslant 5$) and hexagonal N-wall ($1 \leqslant N \leqslant 2$).

(This figure is in colour only in the electronic version)

Table 1. Energy $\widehat{E}$ and cell size L for the square N-wall.

N	$\widehat{E}$	L
1	1.068	4.25
2	1.053	4.47
3	1.048	4.54
4	1.046	4.58
5	1.044	4.61

cell will contain a multiple of $4N$ half-skyrmions, and therefore its topological charge Q will be a multiple of $2N$; this indeed turns out to be the case. As mentioned above, the walls do not merge, but retain their identity; the location of each wall can be determined by looking at the locus where $\Phi_4 = 0$.

The simplest case to describe is the square one, with $L_y = L_x = L$; our results for $1 \leqslant N \leqslant 5$ are summarized in table 1, which gives the energy-per-charge $\widehat{E}$, and the optimal value of L, for each N. Pictures of the $N = 1$ and $N = 2$ cases are presented in figure 1, together with a plot of the energy data in table 1. Let us first comment on the data. The normalized energy $\widehat{E}$ of the square N-wall is surprisingly close to having a $1/N$-dependence (although there is no obvious reason why this should be so), and extrapolating on this basis gives $\widehat{E} \approx 1.039$ in the $N \to \infty$ limit. This is very close to the energy of the (triply-periodic) Skyrme crystal, a cubic array in which each fundamental cube contains eight half-skyrmions:

its energy-per-charge, computed using the method described above, is $\widehat{E} = 1.038$. Further support for the claim that the square N-wall tends to the Skyrme crystal as $N \to \infty$ comes from looking at the symmetries of the field $\mathbf{\Phi}$. These include, for example, the translations

$$\begin{aligned} x \mapsto x + \tfrac{1}{2}L_x &\Rightarrow (\Phi_1, \Phi_2, \Phi_3, \Phi_4) \mapsto (-\Phi_1, -\Phi_2, \Phi_3, \Phi_4), \\ y \mapsto y + \tfrac{1}{2}L_y &\Rightarrow (\Phi_1, \Phi_2, \Phi_3, \Phi_4) \mapsto (\Phi_1, -\Phi_2, -\Phi_3, \Phi_4), \\ z_p \mapsto z_{p+1} \quad &\Rightarrow \quad (\Phi_1, \Phi_2, \Phi_3, \Phi_4) \mapsto (\Phi_1, -\Phi_2, \Phi_3, -\Phi_4), \end{aligned}$$

where the third translation (in z) denotes moving from the pth wall to the $(p+1)$st wall. These are exactly the same as the translation symmetries of the Skyrme crystal [2]. The values for the optimal cell length $L = L_x = L_y$ are consistent with their approaching $L = 4.7$ as $N \to \infty$, this being the cell size of the Skyrme crystal (and similarly the distance between each parallel pair of walls is approximately $4.7/2$, as one would expect).

Each three-dimensional plot in figure 1 is an isosurface of the energy density $\mathcal{E}$, namely where $\mathcal{E}$ equals 0.6 times its maximum value. For the square case, the plots are over four fundamental cells. One clearly sees square arrays of half-skyrmions. Observe that, for $N = 2$, the half-skyrmions are aligned in the z-direction; the same is true for $N > 2$.

Let us turn now to the case of hexagonal symmetry. For ease of computation, we follow the same scheme as in [7], namely taking $L_y = \sqrt{3}L_x$ and fitting two fundamental parallelograms into the corresponding rectangle. Each such rectangle, of each wall, contains eight half-skyrmions, as is seen in the hexagonal 2-wall picture of figure 1. For $N = 1$, the energy of the hexagonal arrangement is $\widehat{E} \approx 1.062$, less than that of the corresponding square case [7], but for $N \geqslant 2$, the hexagonal arrangement is less efficient than the square one, and (depending on the values of L_x and L_y) it is either a local minimum of the energy functional or it is unstable. There is a local minimum hexagonal 2-wall solution with energy $\widehat{E} \approx 1.055$, which is only very slightly (less than 0.2%) higher than that of the square 2-wall. Its energy density is depicted in figure 1; one feature to note is that the two walls are not aligned in the z-direction, but are offset. If L_x and L_y are allowed to change so that the relation $L_y = \sqrt{3}L_x$ no longer holds, then this solution becomes unstable and changes into the square 2-wall.

An isolated skyrmion of charge $Q \geqslant 3$ typically has a polyhedral shell structure, analogous to carbon fullerenes, and it may be viewed as constructed from a section of the hexagonal 1-wall (graphene), with the insertion of defects to create a spherical shell [2, 7]. There has also been an investigation [9] of the possibility of constructing skyrmions as multi-walled spherical shells, with the 'shell material' consisting of a double or triple wall. For the cases that were examined in [9], either the walls coalesced, or one obtained a structure which resembled a shell-like part of the Skyrme crystal. The findings reported above are consistent with this; in particular, multiple hexagonal walls appear to be rather unstable, and therefore unsuitable for constructing shells. But it does raise the possibility of stable high-charge skyrmions constructed as shells of square multi-wall material, or equivalently as hollow chunks of Skyrme crystal, and this would be worth investigating further.

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